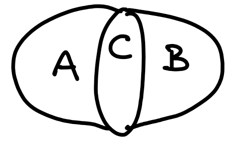
Calculation methods: excision & Hurewicz

π_n is much harder to compute than H_n because excision doesn't work in general! However, there's still something.



$X = A \cup B$, $A \cap B = C$
(X,B) vs. (A,C)?

Thm: $X = A \cup B$ CW-complex, A, B subcomplexes, $A \cap B = C$ nonempty connected. If (A,C) is m-connected and (B,C) is n-connected then the inclusion map induces $\pi_i(A,C) \to \pi_i(X,B)$ isom. for $i < m+n$, surjection for $i = m+n$.

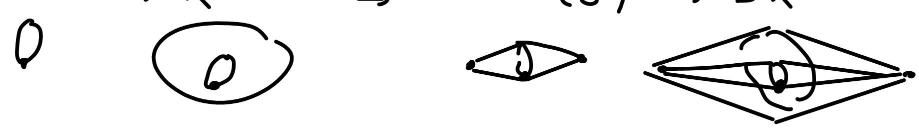
Note: up to $i<n$ is not surprising since we've seen before that, up to replacing by CW-approximations, can assume $B \setminus C$ only has cells of dim $\geq n+1$.

Corollary: Freudenthal suspension theorem

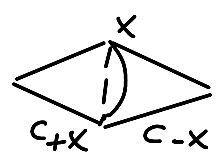
The suspension map $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is an iso for $i < 2n-1$, surjection for $i = 2n-1$.

More generally the same holds for $\pi_i(X) \to \pi_{i+1}(SX)$ whenever X is an $(n-1)$-connected CW-complex.

(suspension map: $S^i \longrightarrow X \Rightarrow S^{i+1} = S(S^i) \to SX$


)

Pf: write $SX = \underbrace{C_+X \cup C_-X}_{\text{cones}}$



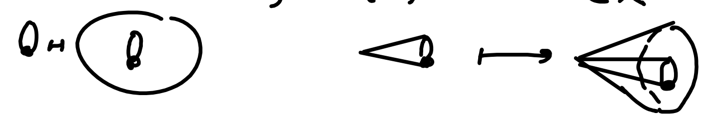
intersecting along X.

Then suspension map $\equiv$ $\pi_i(X) \xrightarrow{\sim} \pi_{i+1}(C_+X, X) \xrightarrow{i_*} \pi_{i+1}(SX, C_-X) \cong \pi_{i+1}(SX)$

inverse to ∂ map: retract C_-X to pt ✓

from l.e.s. $\dots \to \underset{\overset{\shortparallel}{0}\ \text{(contractible)}}{\pi_{i+1}(C_+X)} \to \pi_{i+1}(C_+X, X) \underset{\cong}{\xrightarrow{\partial}} \pi_i(X) \to \underset{\overset{\shortparallel}{0}}{\pi_i(C_+X)} \to \dots$

i.e. "cone" map $S^i \to X \Rightarrow C(S^i) = D^{i+1} \to CX$



X $(n-1)$-connected $\Rightarrow (CX, X)$ is n-connected, so i_* $\begin{cases} \text{iso} & \text{for } i+1 < 2n \\ \text{surj.} & \text{for } i+1 = 2n \end{cases}$ ▲

Corollary: $\forall n \geq 1$, $\pi_n(S^n) \simeq \mathbb{Z}$ gen. by identity map. [see also HW 1].

Pf: Freudenthal $\Rightarrow$ suspending induces $\pi_1(S^1) \twoheadrightarrow \pi_2(S^2) \xrightarrow{\sim} \pi_3(S^3) \xrightarrow{\sim} \ldots$ ($\pi_1(S^1) = \mathbb{Z}$)

$\Rightarrow \pi_n(S^n)$ for $n \geq 2$ is cyclic (finite or infinite).

$f_k = k \cdot id$ is a map of degree k, ie. $f_{k*}[S^n] = k[S^n] \in H_n(S^n) \simeq \mathbb{Z}$

so f_k pairwise non-homotopic, $\pi_n(S^n) = \mathbb{Z}$. ▲

Note however $\pi_3(S^2) = \mathbb{Z}$ (gen^d by Hopf map $S^3 \to S^2$)

$(z_1, z_2) \mapsto (z_1 : z_2) \in \mathbb{CP}^1 = \mathbb{C} \cup \infty = S^2$

st. $|z_1|^2 + |z_2|^2 = 1$

$\downarrow$

$\pi_4(S^3) \simeq \pi_5(S^4) \simeq \ldots \simeq \mathbb{Z}/2$.

($\pi_{n+k}(S^n)$ stabilizes for $n \geq k+2$!)

Proof of Thm: successive cases of increasing generality:

- Case 1: - assume $A = C \cup (m+1)$-cells e_α^{m+1}
  $B = C \cup$ single $(n+1)$-cell e^{n+1}.

(a) To show surjectivity of $\pi_i(A,C) \xrightarrow[i_*]{} \pi_i(X,B)$ for $i \leq m+n$:

let $f : (I^i, \partial I^i, J_i) \to (X, B, x_0)$ — want to push f away from e^{n+1}?

I^i compact $\Rightarrow$ image of f is compact, so meets only finitely many of e_α^{m+1}'s.

PL-approximation lemma: can homotope f so $\exists$ simplices $\Delta_\alpha^{m+1} \subset \text{int}(e_\alpha^{m+1})$, $\Delta^{n+1} \subset \text{int}(e^{n+1})$

so that $f^{-1}(\Delta_\alpha^{m+1})$, $f^{-1}(\Delta^{n+1})$ finite unions of convex polyhedra on which f is PL. (on each, f is a linear map $\mathbb{R}^i \to \mathbb{R}^{m+1}$ or $\mathbb{R}^{n+1}$); can assume the linear maps are surjective (else take smaller Δ_α^{m+1}, Δ^n to avoid image of lower-rank maps).

Key observation: for $q \in \Delta^{n+1}$, $f^{-1}(q)$ = finite union of convex polyhedra of dim $\leq i-n-1$.

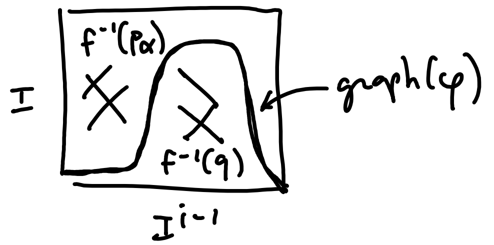
$p_\alpha \in \Delta_\alpha^{m+1}$, $f^{-1}(p_\alpha)$ = ——— " ——— $\leq i-m-1$.

Those are of course mutually disjoint, but we can do better - observe

$i \leq m+n \Rightarrow (i-n-1) + (i-m-1) < i-1$. So if we choose q, p_α generically, the images of these polyhedra under $\pi : I^i \to I^{i-1}$ (forget last coord.) are disjoint.

(specifically, choose $p_\alpha \in \Delta_\alpha^{m+1} \setminus f(\pi^{-1}(\pi(f^{-1}(q))))$ $\leftarrow$ $\cup$ polyhedra dim $\leq i-n \leq m$).

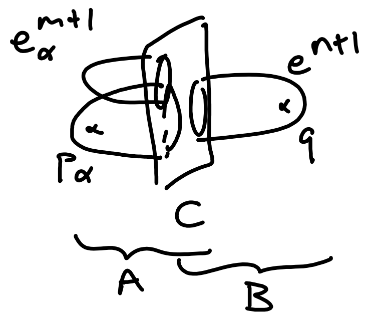
Hence, Lemma: If $i \le m+n$, $\exists\, p_\alpha \in \Delta_\alpha^{m+1}$, $q \in \Delta^{n+1}$ and $\varphi : I^{i-1} \to [0,1)$ s.t. $\varphi = 0$ on $\partial(I^{i-1})$, $f^{-1}(q)$ lies below graph(φ), $f^{-1}(p_\alpha)$ lies above graph(φ) $\forall \alpha$.



This allows us to excise the portion of f below graph(φ) by a homotopy: let f_t = restriction of f to region above graph$(t\varphi)$ $\quad 0 \le t \le 1$ (identifying it with I^{i-1})

By construction:
- $\forall t$, $f_t(I^{i-1})$ is disjoint from $P = \bigcup_\alpha \{p_\alpha\}$
- $f_1(I^i)$ is disjoint from $Q = \{q\}$

so: we didn't quite prove yet that f is homotopic among maps to (X,B) to a map to (A,C), but we did almost as well: we proved it's homotopic among maps to $(X, X-P)$ to a map to $(X-Q, X-(P\cup Q))$



These are homotopy equivalent (collapse $e_\alpha^{m+1} - \{p_\alpha\}$ to its ∂)
— " — $e^{n+1} - \{q\}$ — " —

so in the commutative diagram of inclusion maps

$$\begin{array}{ccc} \pi_i(A,C) & \xrightarrow{i_*} & \pi_i(X,B) \ni [f] \\ \downarrow\wr & & \downarrow\wr \\ \pi_i(X-Q, X-Q-P) & \longrightarrow & \pi_i(X, X-P) \end{array}$$

in lower right group, $[f] = [f_1]$ where f_1 comes from lower-left $\Rightarrow [f] \in \operatorname{im}(i_*)$.

(b) for injectivity: assume $f_0, f_1 : (I^i, \partial I^i, J_i) \to (A, C, x_0)$ $(i < m+n)$ represent same element in $\pi_i(X,B)$: then $\exists$ homotopy $F : (I^i, \partial I^i, J_i) \times [0,1] \to (X, B, x_0)$.
Deform F using PL approximation lemma as before; as above, can find $q \in e^{n+1}$, $p_\alpha \in e_\alpha^{m+1}$, and a function $\varphi : I^{i-1} \times [0,1] \to [0,1)$, $\varphi_{|I^{i-1} \times \{0,1\}} = 0$, whose graph separates $F^{-1}(q)$ from $\bigcup_\alpha F^{-1}(p_\alpha)$.

(Note: the dimension condition is now $i+1 \le m+n$, ie. $i < m+n$).

As before this allows us to excise $F^{-1}(q)$ from the domain of F, ie. deform F to a homotopy between f_0 & f_1 among maps

$(I^i, \partial I^i, J_i) \to (X \setminus Q,\ X \setminus (P \cup Q),\ x_0)$ (which retracts onto (A, C, x_0)).

hence f_0, f_1 represent the same element of $\pi_i(A, C, x_0)$.

<u>Case 2:</u> $A = C \cup (m+1)$-cells e_α^{m+1} as in case 1

$B = C \cup$ cells of dim. $\geq n+1$.

Surj: Any $f: (I^i, \partial I^i, J_i) \to (X, B, x_0)$ hits only finitely many cells (by compactness), and using case 1 repeatedly we can push it off the cells of $B - C$ one at a time (starting with highest-dim. cells).

Inj: similarly for $F: (I^i, \partial I^i, J_i) \times [0,1] \to (X, B, x_0)$ ....

<u>Case 3:</u> $A = C \cup$ cells of dim. $\geq m+1$

$B = C \cup$ cells of dim $\geq n+1$ as in case 2.

By cellular approx., can ignore cells of dim $> m+n+1$ in A (don't affect π_i, $i \le m+n$)

Let $A_k = C \cup$ cells of dim $\le k$, $X_k = A_k \cup B$

$$\begin{array}{ccc} A_k & & X_k \\ \cap & & \cap \\ A & & X \end{array}$$

Prove result for $\pi_i(A_k, C) \to \pi_i(X_k, B)$ by induction on k starting at $k = m+1 \equiv$ case 2 and ending at $k = m+n+1$.

Look at l.e.s. in rel. homotopy for triples (A_k, A_{k-1}, C) and (X_k, X_{k-1}, B):

$$\begin{tikzcd}
\pi_{i+1}(A_k, A_{k-1}) \arrow[r] \arrow[d, "i_*"] & \pi_i(A_{k-1}, C) \arrow[r] \arrow[d, "i_*"] & \pi_i(A_k, C) \arrow[r] \arrow[d] & \pi_i(A_k, A_{k-1}) \arrow[r] \arrow[d] & \pi_{i-1}(A_{k-1}, C) \arrow[d] \\
\pi_{i+1}(X_k, X_{k-1}) \arrow[r] & \pi_i(X_{k-1}, B) \arrow[r] & \pi_i(X_k, B) \arrow[r] & \pi_i(X_k, X_{k-1}) \arrow[r] & \pi_{i-1}(X_{k-1}, B)
\end{tikzcd}$$

for $i < m+n$: iso by case 2   iso by induction   iso by case 2   iso by induction

$\Rightarrow$ by five lemma, middle map is iso. Conclude by induction

(for $i = m+n$, get surjection by one half of five lemma)

(for $i = 1$, argue directly instead).

<u>general case:</u> use CW-approximation to replace (A, C) and (B, C) by homotopy equivalent CW-pairs st. all cells have dim $\geq m+1$ resp. $n+1$, ie. reduce to case 3.

(Since h.e. $(A, C) \simeq (A', C)$ and $(B, C) \simeq (B', C)$ are id on C, fit together to $A' \cup B' \simeq A \cup B$). $\blacktriangle$

Examples: • we've seen above that $\pi_n(S^n) \simeq \mathbb{Z}$. In fact this lets us calculate $\pi_n(\bigvee_\alpha S^n_\alpha) \cong \bigoplus_\alpha \pi_n(S^n_\alpha) = \bigoplus_\alpha \mathbb{Z}$ (for $n \geq 2$)

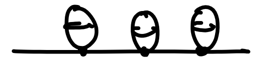
Indeed, for a finite collection, $\prod_\alpha S^n_\alpha = (\bigvee_\alpha S^n_\alpha) \cup (\text{cells of dim} \geq 2n)$

so $\pi_n(\bigvee_\alpha S^n_\alpha) \simeq \pi_n(\prod_\alpha S^n_\alpha) = \prod_\alpha \pi_n(S^n_\alpha)$ ✓ only affects π_i, $i \geq 2n-1$.

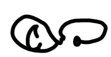
For infinite collection, recall any map $S^n \to \bigvee_\alpha S^n_\alpha$ or any homotopy only hits finitely many of the S^n_α, so get $\bigoplus_\alpha \pi_n(S^n_\alpha)$.

• for $n \geq 2$, $\pi_n(S^1 \vee S^n) =$ free abelian gp w/ countably ∞ generators

indeed $\pi_n(S^1 \vee S^n) \simeq \pi_n(\text{univ. cover})$, but univ. cover $\simeq$ h.e. $\bigvee_\infty S^n$



Note $\pi_1(S^1 \vee S^n) = \mathbb{Z}$. action is non trivial:

generator acts by  to  = next generator, ... so in fact

id S^n

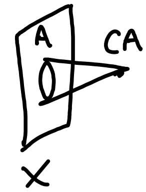
$\pi_n(S^1 \vee S^n) \simeq \mathbb{Z}[t, t^{-1}]$ as a module over $\mathbb{Z}[\pi_1] = \mathbb{Z}[t^{\pm 1}]$.

Friday 2/10

• Another corollary of excision:

Prop: If a CW-pair (X,A) is r-connected and A is s-connected, $r, s \geq 0$ then the maps $\pi_i(X,A) \longrightarrow \pi_i(X/A)$ induced by quotient map $X \to X/A$ are isos. $i \leq r+s$ surjection $i = r+s+1$.

Pf: $X \cup CA$ attach cone on A along A



CA contractible subcomplex

$\Rightarrow X \cup CA \xrightarrow[\text{quotient}]{} (X \cup CA)/CA = X/A$ is a homotopy equivce.

Also: A s-connected $\Rightarrow (CA, A)$ $s+1$-connected (since l.e.s $\Rightarrow \pi_{i+1}(CA,A) \xrightarrow[\simeq]{\partial} \pi_i(A)$)

Excision $\Rightarrow$ inclusion induces $\pi_i(X,A) \xrightarrow{i_*} \pi_i(X \cup CA, CA)$ isom. for $i \leq r+s$, surj. for $i = r+s+1$

$$\begin{tikzcd}
\pi_i(X,A) \arrow[r, "i_*"] \arrow[dr, "\text{quotient map}"'] & \pi_i(X \cup CA, CA) \arrow[d, "\text{quotient}", "\simeq"'] \\
& \pi_i(X/A)
\end{tikzcd}$$

◢

Example: <u>Eilenberg-MacLane spaces</u>

<u>Def.</u> a $K(G,n)$ is a CW-complex K s.t. $\pi_n(K) \cong G$
$\pi_i(K) = 0$ for $i \neq n$.

<u>Construction</u>: for $n \geq 2$ and G any abelian group:

- first build an $(n-1)$ connected CW-complex X with $\pi_n(X) \cong G$:
  start w/ a presentation of G by generators & relations, ie. $G = \left(\bigoplus_\alpha \mathbb{Z}\right)/\underset{\text{subgp.}}{H}$
  → consider a wedge of spheres $\bigvee_\alpha S^n_\alpha$, one sphere for each generator ($\Rightarrow \pi_n = \bigoplus_\alpha \mathbb{Z}$).
  → let $\varphi_\beta : S^n \to \bigvee_\alpha S^n_\alpha$ representing generators of $H = \mathrm{Ker}\left(\bigoplus_\alpha \mathbb{Z} \twoheadrightarrow G\right)$ (ie relations).
  attach $(n+1)$-cells e^{n+1}_β along φ_β to get a CW-complex X.
  → X is $(n-1)$-conn. since only contains n-cells & $(n+1)$-cells
  → l.e.s. of pair $(X, \bigvee_\alpha S^n_\alpha)$:

$$\cdots \to \pi_{n+1}\left(X, \bigvee_\alpha S^n_\alpha\right) \xrightarrow{\partial} \pi_n\left(\bigvee_\alpha S^n_\alpha\right) \to \pi_n(X) \to \pi_n\left(X, \bigvee_\alpha S^n_\alpha\right)$$

$\cong$ by last proposition; $\pi_n\left(X, \bigvee_\alpha S^n_\alpha\right) = 0$ (cellular approx.) on last prop.

$$\pi_{n+1}\left(X/\bigvee_\alpha S^n_\alpha\right) = \pi_{n+1}\left(\bigvee_\beta S^{n+1}_\beta\right) = \bigoplus_\beta \mathbb{Z}$$

so $\pi_{n+1}(X, \bigvee_\alpha S^n_\alpha)$ free w/ basis the charact. maps of cells e^{n+1}_β

By constr., the image of the map to $\pi_n(\bigvee_\alpha S^n_\alpha) = \bigoplus_\alpha \mathbb{Z}$ is exactly H.

Hence $\pi_n(X) \cong \left(\bigoplus_\alpha \mathbb{Z}\right)/H = G$.

- Next kill $\pi_{n+1}(X)$ by attaching $(n+2)$-cells along its generators (w/out modifying π_n)
  then kill $\pi_{n+2}(X)$ by attaching $(n+3)$-cells
  and so on --- to get a $K(G,n)$. ∎

(For $n=1$ & G any group, not necess abelian:
similarly, build a 2-dim. CW complex with $\pi_1 \cong G$ by taking a wedge of S^1's for generators of G and attaching 2-cells along relations
then kill $\pi_2, \pi_3 \ldots$ by attaching higher dim cells)

Ex: • S^1 is $K(\mathbb{Z},1)$, $T^n = (S^1)^n$ is $K(\mathbb{Z}^n,1)$ } $\pi_1 = \dots$
$\mathbb{RP}^\infty$ is $K(\mathbb{Z}/2,1)$ } & univ cover contractible!

• $\mathbb{CP}^\infty$ is $K(\mathbb{Z},2)$ (will see).

Ex: can build X with $\pi_n(X) = G_n$ arbitrary: take $X = \prod_n K(G_n,n)$.

Prop: Any two $K(G,n)$ CW-complexes are homotopy equivalent.

Pf: we'd like to use Whitehead's thm, but need to make sure the iso. of π_n is induced by some actual map. Useful lemma:

Lemma: $X = (\bigvee_\alpha S^n_\alpha) \cup \bigcup_\beta e^{n+1}_\beta$ (as in above construction), $n \geq 1$
then $\forall$ homomorphism $\psi: \pi_n(X) \to \pi_n(Y)$, $\exists f: X \to Y$ st. $f_* = \psi$.

Pf: Recall $\pi_n(X) = (\text{free gp gen by } S^n_\alpha) / (\text{subgroup gen. by } \varphi_\beta: S^n \to \bigvee_\alpha S^n_\alpha)$

• map base point $\mapsto$ base point.

• over S^n_α, let f = a map representing $\psi([i_\alpha]) \in \pi_n(Y)$
$[i_\alpha]$ = class in $\pi_n(X)$ rep by incl. of S^n_α.

• to extend f over cell e^{n+1}_β with attaching map φ_β, need to know that $f \circ \varphi_\beta : S^n \xrightarrow{\varphi_\beta} \bigvee_\alpha S^n_\alpha \xrightarrow{f} Y$ is nullhomotopic - true since $f_*[\varphi_\beta] = \psi([\varphi_\beta]) = \psi(0) = 0$.
Hence extend to $f: X \to Y$

• by constr. $f_*([i_\alpha]) = \psi([i_\alpha])$ $\forall \alpha$, $[i_\alpha]$ generate $\pi_n(X)$, so $f_* = \psi$. ∎

So: consider the construction above building a $K(G,n)$ K by attaching higher dim. handles to $X = (\bigvee_\alpha S^n_\alpha) \cup (\bigcup_\beta e^{n+1}_\beta)$ to kill its $\pi_{n+1}, \pi_{n+2}, \dots$
and let K' = any other $K(G,n)$.
By lemma, $\exists$ map $f: X \to K'$ realizing isomorphism of $\pi_n(X) \simeq \pi_n(K') \simeq G$.
To extend f to rest of K: for each $(n+2)$-cell e^{n+2} w/ attaching map $\varphi: S^{n+1} \to X$, $f \circ \varphi$ is nullhomotopic in K' since $\pi_{n+1}(K') = 0$, hence f extends across cell.
Continue over all higher dim! cells of K, and obtain $f: K \to K'$.
By Whitehead's thm, f is a homotopy eq$\underline{\text{ce}}$. ∎